

Decay of Almost Periodic Solutions of Anisotropic Degenerate Parabolic-Hyperbolic Equations

Hermano Frid

Dedicated to Helge Holder on his 60th birthday.

Abstract. We prove the well-posedness and decay of Besicovitch almost periodic solutions for nonlinear degenerate anisotropic hyperbolic-parabolic equations. We also investigate the case where the diffusion term is given by a non-degenerate nonlinear $d'' \times d''$ diffusion matrix and the complementary d' components of flux-function form a non-degenerate flux in $\mathbb{R}^{d'}$, with $d' + d'' = d$. For this special case we prove that the strong trace property at the initial time holds, which allows to require the assumption of the initial data only in a weak sense, as well as the continuity in time of the solution with values in $L^1_{\text{loc}}(\mathbb{R}^d)$.

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1. Introduction

We address the problem of the decay to the mean-value of L^∞ Besicovitch almost periodic solutions to nonlinear degenerate anisotropic hyperbolic-parabolic equations. Consider the Cauchy problem

$$\partial_t u + \nabla_x \cdot \mathbf{f}(u) = \nabla_x \cdot (A(u) \nabla_x u), \quad x \in \mathbb{R}^d, \quad t > 0, \quad (1.1)$$

$$u(0, x) = u_0, \quad x \in \mathbb{R}^d, \quad (1.2)$$

where $\mathbf{f} = (f_1, \dots, f_d)$, $A(u) = (a_{ij}(u))_{i,j=1}^d$, with $f_i(u), a_{ij}(u) : \mathbb{R} \rightarrow \mathbb{R}$ smooth functions. $A(u)$ is a symmetric non-negative matrix and so we may write

$$a_{ij}(u) = \sum_{k=1}^d \sigma_{ik}(u) \sigma_{jk}(u), \quad (1.3)$$

with $\sigma_{ij}(u) : \mathbb{R} \rightarrow \mathbb{R}$ smooth functions, that is, $(\sigma_{ij}(u))_{i,j=1}^d$ is the square root of $A(u)$. We assume to begin with that $u_0 \in L^\infty(\mathbb{R}^d)$.

We impose the following non-degeneracy condition on $(\mathbf{f}(u), A(u))$: For any $(\tau, \kappa) \in \mathbb{R}^{d+1}$ with $\tau^2 + \kappa^2 = 1$, we have

$$\mathcal{L}^1\{\xi \in \mathbb{R} : |\xi| \leq \|u_0\|_\infty, \tau + \mathbf{a}(\xi) \cdot \kappa = 0, \kappa^\top A(\xi) \kappa = 0\} = 0. \quad (1.4)$$

In this paper, we are concerned with the large-time behavior of entropy solutions of (1.1),(1.2) with initial function u_0 satisfying

$$u_0 \in L^\infty(\mathbb{R}^d) \cap \text{BAP}(\mathbb{R}^d). \quad (1.5)$$

Here, $\text{BAP}(\mathbb{R}^d)$ demotes the space of the Besicovitch almost periodic functions (with exponent $p = 1$), which can be defined as the completion of the space of trigonometric polynomials, i.e., finite sums $\sum_\lambda a_\lambda e^{2\pi i \lambda \cdot x}$ ($i = \sqrt{-1}$ is the purely imaginary unity) under the semi-norm

$$N_1(g) := \limsup_{R \rightarrow \infty} \frac{1}{R^d} \int_{C_R} |g(x)| dx,$$

where, for $R > 0$,

$$C_R := \{x \in \mathbb{R}^d : |x|_\infty := \max_{i=1, \dots, d} |x_i| \leq R/2\}.$$

We observe that the semi-norm N_1 is indeed a norm over the trigonometric polynomials, so the referred completion through it is a well defined Banach space. Equivalently, the space $\text{BAP}(\mathbb{R}^d)$ is also the completion through N_1 of the space of uniform (or Bohr) almost periodic functions, $\text{AP}(\mathbb{R}^d)$, which the defined as the closure in the sup-norm of the trigonometric polynomials.

We begin by stating the definition of entropy solution for (1.1),(1.2), which is in part motivated by [9]. We use the normal trace property of L^2 -divergence measure fields (see, e.g., [6, 7]).

Definition 1.1 *An entropy solution for (1.1),(1.2), with $u_0 \in L^\infty(\mathbb{R}^d)$, is a function $u(t, x) \in L^\infty((0, \infty) \times \mathbb{R}^d)$ such that*

(i) *(Regularity) For any $R > 0$, we have*

$$\sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \in L^2((0, \infty) \times C_R),$$

$$\text{for } k = 1, \dots, d, \text{ for } \beta_{ik}(u) = \int^u \sigma_{ik}(v) dv. \quad (1.6)$$

(ii) *(Chain Rule) For any function $\psi \in C_0(\mathbb{R})$ with $\psi(u) \geq 0$ and any $k = 1, \dots, d$ the following chain rule holds:*

$$\sum_{i=1}^d \partial_{x_i} \beta_{ik}^\psi(u) = \sqrt{\psi(u)} \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \in L^2((0, \infty) \times C_R),$$

$$\text{for } k = 1, \dots, d, \text{ for } (\beta_{ik}^\psi)' = \sqrt{\psi} \beta_{ik}', \quad (1.7)$$

for any $R > 0$.

(iii) (*Entropy Inequality*) For any convex C^2 function $\eta : \mathbb{R} \rightarrow \mathbb{R}$, and $\mathbf{q}'(u) = \eta'(u)\mathbf{f}(u)$, $r'_{ij}(u) = \eta'(u)a_{ij}(u)$, we have

$$\partial_t \eta(u) + \nabla_x \cdot \mathbf{q}(u) - \sum_{ij=1}^d \partial_{x_i x_j}^2 r_{ij}(u) \leq -\eta''(u) \sum_{k=1}^d \left(\sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right)^2, \quad (1.8)$$

in the sense of distributions in $(0, \infty) \times \mathbb{R}^d$, and

$$\eta(u(t, x))|_{\{t=0\}} = \eta(u_0(x)), \quad (1.9)$$

in the sense of the normal trace of the L^2 -divergence measure field

$$\left(\eta(u), \mathbf{q}(u) - \left(\sum_{j=1}^d \partial_{x_j} r_{ij}(u) \right)_{i=1}^d \right).$$

Remark 1.1 We remark that condition (iii) in the Definition 1.1 implies that for all $k \in \mathbb{R}$ we have

$$\begin{aligned} & \partial_t |u(t, x) - k| + \nabla_x \cdot \text{sgn}(u(t, x) - k)(\mathbf{f}(u) - \mathbf{f}(k)) \\ & - \sum_{i,j=1}^d \partial_{x_i x_j}^2 \text{sgn}(u(t, x) - k)(A_{ij}(u) - A_{ij}(k)) \geq 0, \end{aligned} \quad (1.10)$$

where $A'_{ij}(u) = a_{ij}(u)$, in the sense of distributions in $(0, \infty) \times \mathbb{R}^d$.

Remark 1.2 We also remark that (1.9), valid for all C^2 convex η implies, for any $R > 0$,

$$\lim_{t \rightarrow 0+} \int_{C_R} |u(t, x) - u_0(x)| dx = 0, \quad (1.11)$$

as essentially follows from theorem 4.5.1 in [11] (see also [15]) which establishes that (1.9) implies

$$\lim_{t \rightarrow 0+} \int_{\mathbb{R}^d} \eta(u(t, x)) \phi(x) dx = \int_{\mathbb{R}^d} \eta(u_0(x)) \phi(x) dx,$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$, which by a well known convexity argument implies (1.11).

Remark 1.3 Take $\eta(u) = \frac{1}{2}u^2$ in (1.8) and as test function $\phi_R(x)\chi_\nu(t)$, with $\phi_R \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \phi_R(x) \leq 1$, for all $x \in \mathbb{R}^d$, $\phi_R(x) = 1$, for $|x| \leq R$, $\phi_R(x) = 0$, for $|x| \geq R+1$, and $\|D^\alpha \phi_R\|_\infty \leq C$, $|\alpha| \leq 2$, for some $C > 0$ independent of R , and $\chi_\nu(t) = \theta(t - t_0) - \theta(t - t_1)$, with

$$\theta_\nu(t) = \int_0^t \delta_\nu(s) ds = \int_0^{\nu t} \sigma(s) ds, \quad \delta_\nu(s) = \nu \sigma(\nu s),$$

with $\sigma \in C_0^\infty(\mathbb{R})$, $\text{supp } \sigma \subset [0, 1]$, $\sigma \geq 0$, $\int_{\mathbb{R}} \sigma(s) ds = 1$. Then, sending $\nu \rightarrow \infty$ we deduce that for some constant $C > 0$, independent of R , we have, for all $t > 0$,

$$\int_0^t \int_{C_R} \sum_{k=1}^d \left(\sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right)^2 dx dt \leq C(R+1)^d + Ct(R+1)^{d-1}. \quad (1.12)$$

In particular, for any $t > 0$,

$$\limsup_{R \rightarrow \infty} R^{-d} \int_0^t \int_{C_R} \sum_{k=1}^d \left(\sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right)^2 dx dt \leq C. \quad (1.13)$$

For any $g \in \text{BAP}(\mathbb{R}^d)$, its mean value $M(g)$, defined by

$$M(g) := \lim_{R \rightarrow \infty} R^{-d} \int_{C_R} g(x) dx,$$

exists (see, e.g., [2]). The mean value $M(g)$ is also denoted by $\mathbf{f}_{\mathbb{R}^d} g dx$. Also, the Bohr-Fourier coefficients of $g \in \text{BAP}(\mathbb{R}^d)$

$$a_\lambda = M(g e^{-2\pi i \lambda \cdot x}),$$

are well defined and we have that the spectrum of g , defined by

$$\text{Sp}(g) := \{\lambda \in \mathbb{R}^g : a_\lambda \neq 0\},$$

is at most countable (see, e.g., [2]). We denote by $\text{Gr}(g)$ the smallest additive subgroup of \mathbb{R}^d containing $\text{Sp}(g)$ (cf. [22], where $\text{Gr}(g)$ was introduced and denoted by $M(g)$).

The first main result of this paper is the following.

Theorem 1.1 *For any $u_0 \in L^\infty(\mathbb{R}^d)$, there exists a unique weak entropy solution $u(t, x)$ of (1.1), (1.2). Moreover, if u_0 satisfies (1.5), then*

$$u \in L^\infty((0, \infty), \text{BAP}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+^{d+1}), \quad (1.14)$$

and $\text{Gr}(u(t, \cdot)) \subset \text{Gr}(u_0)$, for a.e. $t > 0$. Further,

$$\lim_{t \rightarrow +\infty} M(|u(t, \cdot) - M(u_0)|) = 0. \quad (1.15)$$

A particular case of (1.1) is the following

$$\partial_t u + \nabla_x \cdot \mathbf{f}(u) = \nabla_{x''} (B(u) \nabla_{x''} u), \quad x \in \mathbb{R}^d, \quad t > 0, \quad (1.16)$$

where $B(u) = (b_{ij}(u))_{i,j=d'+1}^d$, and $1 \leq d' < d$, so $B(u)$ is a symmetric non-negative $d'' \times d''$ -matrix, $d'' = d - d'$, and $\nabla_{x''} := (\partial_{x_{d'+1}}, \dots, \partial_{x_d})$. Also, instead of (1.4), we now assume the stronger condition: For any $(\tau, \kappa') \in \mathbb{R}^{d'+1}$, with $\tau^2 + \kappa'^2 = 1$, with $\pi_{d'}(\mathbf{f}(u)) = (f_1(u), \dots, f_{d'}(u))$,

$$\mathcal{L}^1\{\xi \in \mathbb{R} : |\xi| \leq \|u_0\|_\infty, \tau + \pi_{d'}(\mathbf{f}(u)) \cdot \kappa' = 0\} = 0, \quad (1.17)$$

$$\mathcal{L}^1\{\xi \in \mathbb{R} : |\xi| \leq \|u_0\|_\infty, \kappa'^\top B(\xi)\kappa' = 0\} = 0. \quad (1.18)$$

Although (1.16) is a particular case of (1.1), under the non-degeneracy conditions (1.17) and (1.18) we may relax (1.9) in Definition 1.1 to

$$u(t, x)|_{\{t=0\}} = u_0(x), \quad (1.19)$$

in the sense of the normal trace of the L^2 divergence-measure field

$$(u, \mathbf{f}(u) - (\underbrace{0, \dots, 0}_{d'}, (\sum_{j=d'+1}^d \partial_{x_j} B_{ij}(u))_{i=d'+1}^d)), \quad B'_{ij}(u) = b_{ij}(u).$$

We call $u(t, x) \in L^\infty((0, \infty) \times \mathbb{R}^d)$ a *weak entropy solution* of (1.16),(1.2) if it satisfies all the corresponding conditions of Definition 1.1 except that instead of (1.9), we now impose the weaker (1.19).

The second result of this paper concerns weak entropy solutions of (1.16),(1.2).

Theorem 1.2 *Let u be weak entropy solution of (1.16),(1.2). Then,*

$$u \in C([0, \infty), L^1_{\text{loc}}(\mathbb{R}^d)).$$

In particular, for any $R > 0$,

$$\lim_{t \rightarrow 0^+} \int_{|x| < R} |u(t, x) - u_0(x)| dx = 0. \quad (1.20)$$

Moreover, if u_0 satisfies (1.5), then

$$u \in C([0, \infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}_+^{d+1}), \quad (1.21)$$

$\text{Gr}(u(t, \cdot)) \subset \text{Gr}(u_0)$, for all $t > 0$, and (1.15) holds.

From Theorem 1.2, we deduce that weak entropy solutions of (1.16),(1.2) are indeed entropy solutions of (1.16),(1.2) in the sense of Definition 1.1, so that Theorem 1.1 applies to them. As we will see in Section 3, the proof of Theorem 1.2 amounts to show the validity of the strong trace property for the solution of (1.16),(1.2).

There is a large literature related with degenerate parabolic equations, being the first important contribution by Vol'pert and Hudjaev in [27]. Uniqueness for the homogeneous Dirichlet problem, for the isotropic case, was only

achieved many years later by Carrillo in [3], using an extension of Kruzhkov's doubling of variables method [18]. The result in [3] was extended to non-homogeneous Dirichlet data by Mascia, Porretta and Terracina in [20]. An L^1 theory for the Cauchy problem for anisotropic degenerate parabolic equations was established by Chen and Perthame [9], based on the kinetic formulation (see [23]), and later also obtained using Kruzhkov's approach in [1, 8] (see also, [17], [13] and the references therein). Decay of almost periodic solutions for general nonlinear systems of conservation laws of parabolic and hyperbolic types was first addressed in [14], as an extension of the ideas put forth in [4]. Only recently the problem of the decay of almost periodic solutions was re-taken, specifically for scalar conservation laws, by Panov in [22], where some elegant ideas were introduced to successfully extend the result in [14] in that specific case.

We give a brief account on the way Theorem 1.1. The part of existence and uniqueness are by now well known and for most of that we just refer to [8], which deals with the case of initial function in $L^1(\mathbb{R}^d)$. Nevertheless, (1.13) is new and of great interest in the case of initial functions in $L^\infty(\mathbb{R}^d)$. For the invariance of the class of L^∞ Besicovitch almost periodic functions with exponent $p = 1$, we use the elegant method of reduction to the periodic case introduced by Panov in [22], combined either with the result on the decay of periodic entropy solutions for nonlinear anisotropic degenerate parabolic-hyperbolic equations of Chen and Perthame in [10]. In the particular case where (1.1) has the particular form (1.16) together with the non degeneracy conditions (1.17) and (1.18), we show an alternative proof of the decay property using the techniques in [14].

This paper is organized as follows. After this Introduction, in Section 2, the proof of Theorem 1.1 is given, split in a number of auxiliary results, starting with Proposition 2.1, followed by four lemmas. In Section 3, the proof of Theorem 1.2 is given. Finally, in Section 4, we give an alternative direct proof of the decay property.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 through a number of auxiliary results and results that establish parts of the its statement.

We begin with a proposition which is central in the whole strategy of reducing to the periodic case as devised in [22]. We will need the following technical lemma of [22], to which we refer for the proof.

Lemma 2.1 *Suppose that $u(x, y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m)$,*

$$E = \{x \in \mathbb{R}^n : (x, y) \text{ is a Lebesgue point of } u(x, y) \text{ for a.e. } y \in \mathbb{R}^m\}.$$

Then E is a set of full measure and $x \in E$ is a common Lebesgue point of the functions $I(x) = \int_{\mathbb{R}^m} u(x, y) \rho(y) dy$, for all $\rho \in L^1(\mathbb{R}^m)$.

Proposition 2.1 (mean L^1 -contraction). *Let $u(t, x), v(t, x) \in L^\infty(\mathbb{R}_+^{d+1})$ be two entropy solutions of (1.1), (1.2), with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$. Then for a.e. $0 < t_0 < t_1$*

$$N_1(u(t_1, \cdot) - v(t_1, \cdot)) \leq N_1(u(t_0, \cdot) - v(t_0, \cdot)), \quad (2.1)$$

and also for a.e. $t > 0$,

$$N_1(u(t, \cdot) - v(t, \cdot)) \leq N_1(u_0 - v_0), \quad (2.2)$$

PROOF: We follow closely with the due adaptations the proof of proposition 1.3 in [22]. We first recall that by using the doubling of variables method of Kruzhkov [18], as adapted by Carrillo [3] to the isotropic degenerate parabolic case and [1] to the anisotropic one, we obtain

$$|u-v|_t + \nabla \cdot \text{sgn}(u-v)(\mathbf{f}(u) - \mathbf{f}(v)) \leq \sum_{i,j=1}^d \partial_{x_i x_j}^2 \text{sgn}(u-v)(A_{ij}(u) - A_{ij}(v)) \quad (2.3)$$

in the sense of distributions in \mathbb{R}_+^{d+1} . As usual, we define a sequence approximating the indicator function of the interval $(t_0, t_1]$, by setting for $\nu \in \mathbb{N}$,

$$\delta_\nu(s) = \nu \sigma(\nu s), \quad \theta_\nu(t) = \int_0^t \delta_\nu(s) ds = \int_0^{\nu t} \sigma(s) ds,$$

where $\sigma \in C_0^\infty(\mathbb{R})$, $\text{supp } \sigma \subset [0, 1]$, $\sigma \geq 0$, $\int_{\mathbb{R}} \sigma(s) ds = 1$. We see that $\delta_\nu(s)$ converges to the Dirac measure in the sense of distributions in \mathbb{R} while $\theta_\nu(t)$ converges everywhere to the Heaviside function. For $t_1 > t_0 > 0$, if $\chi_\nu(t) = \theta_\nu(t - t_0) - \theta_\nu(t - t_1)$, then $\chi_\nu \in C_0^\infty(\mathbb{R}_+)$, $0 \leq \chi_\nu \leq 1$, and the sequence $\chi_\nu(t)$ converges everywhere, as $\nu \rightarrow \infty$, to the indicator function of the interval $(t_0, t_1]$. Let us take $g \in C_0^\infty(\mathbb{R}^d)$, satisfying $0 \leq g \leq 1$, $g(y) \equiv 1$ in the cube C_1 , $g(y) \equiv 0$ outside the cube C_k , with $k > 1$. We apply (1.9) to the test function $\varphi = R^{-d} \chi_\nu(t) g(x/R)$, for $R > 0$. We then get

$$\begin{aligned} & \int_0^\infty (R^{-d} \int_{\mathbb{R}^d} |u(t, x) - v(t, x)| g(x/R) dx) (\delta_\nu(t - t_0) - \delta_\nu(t - t_1)) dt \\ & + R^{-d-1} \iint_{\mathbb{R}_+^{d+1}} \text{sgn}(u-v)(\mathbf{f}(u) - \mathbf{f}(v)) \cdot \nabla_y g(x/R) \chi_\nu(t) dx dt \\ & - R^{-d-1} \sum_{i,j=1}^d \iint_{\mathbb{R}_+^{d+1}} \text{sgn}(u-v) \partial_{x_i} (A_{ij}(u) - A_{ij}(v)) \partial_{x_j} g(x/R) \chi_\nu(t) dx dt \geq 0. \end{aligned} \quad (2.4)$$

Define

$$F = \{t > 0 : (t, x) \text{ is a Lebesgue point of } |u(t, x) - v(t, x)| \text{ for a.e. } x \in \mathbb{R}^d\}.$$

As a consequence of Fubini's theorem, F is a set of full Lebesgue measure and by Lemma 2.1 each $t \in F$ is a Lebesgue point of the functions

$$I_R(t) = R^{-d} \int_{\mathbb{R}^d} |u(t, x) - v(t, x)| g(x/R) dx,$$

for all $R > 0$ and all $g \in C_0(\mathbb{R})$. Now we assume $t_0, t_1 \in F$ and take the limit as $\nu \rightarrow \infty$ in (1.10), to get

$$\begin{aligned} I_R(t_1) &\leq I_R(t_0) + R^{-d-1} \iint_{(t_0, t_1) \times \mathbb{R}^d} \operatorname{sgn}(u - v)(\mathbf{f}(u) - \mathbf{f}(v)) \cdot \nabla_y g(x/R) dx dt \\ &\quad - R^{-d-1} \sum_{i,j=1}^d \iint_{(t_0, t_1) \times \mathbb{R}^d} \operatorname{sgn}(u - v) \partial_{x_i}(A_{ij}(u) - A_{ij}(v)) \partial_{x_j} g(x/R) dx dt. \end{aligned} \quad (2.5)$$

Now, we have

$$\begin{aligned} &R^{-d-1} \left| \iint_{(t_0, t_1) \times \mathbb{R}^d} \operatorname{sgn}(u - v)(\mathbf{f}(u) - \mathbf{f}(v)) \cdot \nabla_y g(x/R) dx dt \right| \\ &\leq R^{-1} \|\mathbf{f}(u) - \mathbf{f}(v)\|_\infty \iint_{(t_0, t_1) \times \mathbb{R}^d} |\nabla_y g(y)| dy dt \rightarrow 0, \quad \text{as } R \rightarrow \infty. \end{aligned} \quad (2.6)$$

Also, we have

$$\begin{aligned} &R^{-d-1} \left| \sum_{i,j=1}^d \iint_{R_+^{d+1}} \operatorname{sgn}(u - v) \partial_{x_i}(A_{ij}(u) - A_{ij}(v)) \partial_{x_j} g(x/R) \chi_\nu(t) dx dt \right| \\ &\leq R^{-d-1} \left| \iint_{R_+^{d+1}} \sum_{k,j=1}^d \left(|\beta_{jk}(u)| \left(\left| \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right| \right) \partial_{x_j} g(x/R) \chi_\nu(t) dx dt \right| \right. \\ &\quad \left. + R^{-d-1} \left| \iint_{R_+^{d+1}} \sum_{k,j=1}^d \left(|\beta_{jk}(v)| \left(\left| \sum_{i=1}^d \partial_{x_i} \beta_{ik}(v) \right| \right) \partial_{x_j} g(x/R) \chi_\nu(t) dx dt \right| \right| \\ &\leq CR^{-1} \sum_{k=1}^d \left(R^{-d} \iint_{(t_0, t_1) \times C_{kR}} \left(\left| \sum_{i=1}^d \partial_{x_i} \beta_{ik}(u) \right|^2 + \left| \sum_{i=1}^d \partial_{x_i} \beta_{ik}(v) \right|^2 \right) dx dt \right)^{1/2} \\ &\quad \times \left(\iint_{(t_0, t_1) \times \mathbb{R}^d} |\nabla_y g(y)|^2 dy dt \right)^{1/2} \\ &\longrightarrow 0 \quad \text{as } R \rightarrow \infty, \end{aligned} \quad (2.7)$$

where we have used (1.13). On the other hand, we have

$$N_1(u(t, \cdot) - v(t, \cdot)) \leq \limsup_{R \rightarrow \infty} I_R(t) \leq k^d N_1(u(t, \cdot) - v(t, \cdot)),$$

so taking the limit as $R \rightarrow \infty$ in (2.5), for $t_0, t_1 \in F$, $t_0 < t_1$, we get

$$N_1(u(t_1, \cdot) - v(t_1, \cdot)) \leq k^d N_1(u(t_0, \cdot) - v(t_0, \cdot)),$$

and since $k > 1$ is arbitrary we can make $k \rightarrow 1+$ to get the desired result. Finally, for $t_0 = 0$, we use (1.11) to send $t_0 \rightarrow 0+$ in (2.5) and proceed exactly as we have just done. \square

Since $u_0 \in L^\infty(\mathbb{R}^d)$, we may define $u_{\max} = \sup_{\mathbb{R}^d} u_0(x)$ and $u_{\min} = \inf_{\mathbb{R}^d} u_0(x)$, and we may assume $u_{\max} - u_{\min} > 0$, since for u_0 constant the problem is trivial.

Lemma 2.2 (Uniqueness) *The problem (1.1),(1.2) has at most one entropy solution.*

PROOF: The proof follows through standard arguments (*cf.*, e.g., [27]). So, let $u, v \in L^\infty(\mathbb{R}_+^{d+1})$ be two weak entropy solutions. As in Proposition 2.1, by using the doubling of variables method of Kruzhkov [18], as adapted by Carrillo [3] to the isotropic degenerate parabolic case and [1] to the anisotropic one, we obtain

$$\begin{aligned} & \iint_{\mathbb{R}_+^{d+1}} \{|u - v| \phi_t + \operatorname{sgn}(u - v)(\mathbf{f}(u) - \mathbf{f}(v)) \cdot \nabla \phi \\ & + \sum_{i,j=1}^d \operatorname{sgn}(u - v)(A_{ij}(u) - A_{ij}(v)) \partial_{x_i x_j}^2 \phi\} dx dt \geq 0, \end{aligned} \quad (2.8)$$

for all $0 \leq \phi \in C_0^\infty(\mathbb{R}_+^{d+1})$. We take $\phi(t, x) = \rho(x) \chi_\nu(t)$, where $\rho(x) = e^{-\sqrt{1+x^2}}$ and χ_ν is as in the proof of Proposition 2.1. We observe that

$$\sum_{i=1}^d |\partial_{x_i} \rho(x)| + \sum_{i,j=1}^d |\partial_{x_i x_j}^2 \rho(x)| \leq C \rho(x),$$

for some constant $C > 0$ depending only on d . Hence, making $\nu \rightarrow 0$, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^d} |u(t_1, x) - v(t_1, x)| \rho(x) dx & \leq \int_{\mathbb{R}^d} |u(t_0, x) - v(t_0, x)| \rho(x) dx \\ & + \tilde{C} \int_{t_0}^{t_1} \int_{\mathbb{R}^d} |u(s, x) - v(s, x)| \rho(x) dx dt, \end{aligned}$$

for a.e. $0 < t_0 < t_1$, for some $\tilde{C} > 0$ depending only on \mathbf{f}, A and the dimension d . Therefore, using Gronwall and (1.11), we conclude

$$\int_{\mathbb{R}^d} |u(t, x) - v(t, x)| \rho(x) dx \leq e^{\tilde{C}t} \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \rho(x) dx, \quad (2.9)$$

which gives the desired result. \square

Observe that in the same way we got (2.9) from (2.8), we may get

$$\int_{\mathbb{R}^d} (u(t, x) - v(t, x))_+ \rho(x) dx \leq e^{\tilde{C}t} \int_{\mathbb{R}^d} (u_0(x) - v_0(x))_+ \rho(x) dx, \quad (2.10)$$

from

$$\begin{aligned} & \iint_{\mathbb{R}_+^{d+1}} \{ (u - v)_+ \phi_t + \operatorname{sgn}(u - v)_+ (\mathbf{f}(u) - \mathbf{f}(v)) \cdot \nabla \phi \\ & + \sum_{i,j=1}^d \operatorname{sgn}(u - v)_+ (A_{ij}(u) - A_{ij}(v)) \partial_{x_i x_j}^2 \phi \} dx dt \geq 0, \end{aligned} \quad (2.11)$$

where $(u - v)_+ = \max\{0, u - v\}$ and $\operatorname{sgn}(u - v)_+ = H(u - v)$ where $H(s)$ is the Heaviside function. Taking $v = k$, with $k > \|u_0\|_\infty$, and then reversing the roles of u and v , making $u = k$ and $v = u$, with $k < -\|u_0\|_\infty$, we deduce that

$$|u(t, x)| \leq \|u_0\|_\infty, \quad \text{for a.e. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d. \quad (2.12)$$

Lemma 2.3 (Existence) *There exists an entropy solution to the problem (1.1), (1.2).*

PROOF: We consider the problem (1.1), (1.2) with initial function $u_{0,R}(x) = u_0(x) \chi_{B_R}(x)$, where $B_R = B(0, R)$ is the open ball with radius R centered at the origin. By the existence theorem in [9], which holds for initial data in $L^1(\mathbb{R}^d)$, we obtain an entropy solution $u_R(t, x)$ of (1.1), (1.2)_R. Now, using (2.9), we see that, for a.e. $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}^d} |u_R(t, x) - u_{\tilde{R}}(t, x)| \rho(x) dx & \leq e^{\tilde{C}t} \int_{\mathbb{R}^d} |u_{0,R}(x) - u_{0,\tilde{R}}(x)| \rho(x) dx \\ & \longrightarrow 0, \text{ as } R, \tilde{R} \rightarrow \infty. \end{aligned} \quad (2.13)$$

Therefore, $u_R(t, x)$ converges in $L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^d)$ to a function $u(t, x)$, which satisfies the bound in (2.12) since it holds for all u_R . It is now easy to deduce from the fact that the u_R 's satisfy all conditions of Definition 1.1 that $u(t, x)$

also satisfies all those conditions. We just observe that for the verification of (1.8) from the fact that the u_R 's satisfy (1.8), we use the uniform boundedness in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ of

$$\sum_{k=1}^d \left(\sum_{i=1}^d \partial_{x_i} \beta_{ik}(u_R) \right)^2$$

and Fatou's Lemma. Also, (1.9) is proved by including the initial function in (1.8), with $u(t, x)$ replaced by $u_R(t, x)$, tested against any function in $C_0^\infty(\mathbb{R}^{d+1})$, and taking the limit as $R \rightarrow \infty$, to conclude that (1.9) also holds. \square

In the next lemma, we prove that the solution operator for (1.1), (1.2) take bounded Besicovitch almost periodic functions into bounded Besicovitch almost periodic functions and that $\text{Gr}(u(t, \cdot)) \subset \text{Gr}(u_0(\cdot))$.

Lemma 2.4 *Let $u(t, x)$ be the entropy solution of (1.1), (1.2) with u_0 satisfying (1.5). Let $G_0 = \text{Gr}(u_0)$. Then, $u(t, x) \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}_+^{d+1})$ and $\text{Sp}(u(t, \cdot)) \subset G_0$, for a.e. $t > 0$.*

PROOF: The proof follows by the elegant method of reduction to the periodic case introduced by Panov in [22], more specifically theorems 2.1 and 2.2 in [22]. Here we limit ourselves to indicate the few adaptations that need to be made. The method begins by considering the case where the initial function u_0 is given by a trigonometric polynomial,

$$u_0(x) = \sum_{\lambda \in \Lambda} a_\lambda e^{2\pi i \lambda \cdot x}, \quad (2.14)$$

where $\Lambda = \text{Sp}(u_0) \subset \mathbb{R}^d$ is a finite set. Since u_0 is real we have that $-\Lambda = \Lambda$ and $a_{-\lambda} = \bar{a}_\lambda$, where as usual \bar{z} is the complex conjugate of $z \in \mathbb{C}$. The first observation is that we may find a basis for G_0 , $\{\lambda_1, \cdot, \lambda_m\}$, so that any $\lambda \in G_0$ can be uniquely written as $\lambda = \lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j$, $\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$, and the vectors λ_j are linearly independent over \mathbb{Z} and so also over \mathbb{Q} . Let $J = \{\bar{k} \in \mathbb{Z}^m : \lambda(\bar{k}) \in \Lambda\}$. Then

$$u_0(x) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^m k_j \lambda_j \cdot x}, \quad a_{\bar{k}} := a_{\lambda(\bar{k})}. \quad (2.15)$$

We then have $u_0(x) = v_0(y(x))$, where

$$v_0(y) = \sum_{\bar{k} \in J} a_{\bar{k}} e^{2\pi i \bar{k} \cdot y} \quad (2.16)$$

is a periodic function, $v_0(y + e_i) = v_0(y)$, $i = 1, \dots, m$, e_i the elements of the canonical basis of \mathbb{R}^m , and

$$y(x) = (y_1, \dots, y_m), \quad y_j = \lambda_j \cdot x = \sum_{k=1}^d \lambda_{jk} x_k, \quad \lambda_j = (\lambda_{j1}, \dots, \lambda_{jd}).$$

We then consider the nonlinear degenerate parabolic-hyperbolic equation

$$v_t + \nabla_y \cdot \tilde{\mathbf{f}}(v) = (\mathcal{B} \nabla_y) \cdot (A(v)(\mathcal{B} \nabla_y)v), \quad v = v(t, y), \quad t > 0, \quad y \in \mathbb{R}^m, \quad (2.17)$$

with $\tilde{\mathbf{f}} = (\tilde{f}_1, \dots, \tilde{f}_m)$ and

$$\tilde{f}_j(v) = \lambda_j \cdot \mathbf{f}(v) = \sum_{k=1}^d \lambda_{jk} f_k(v), \quad j = 1, \dots, m, \quad \mathcal{B} = \frac{\partial y}{\partial x}^\top,$$

and

$$\mathcal{B} \nabla_y = \frac{\partial y}{\partial x}^\top \nabla_y = \left(\sum_{j=1}^m \lambda_{j1} \partial_{y_j}, \dots, \sum_{j=1}^m \lambda_{jd} \partial_{y_j} \right).$$

We consider the Cauchy problem for (2.17) with initial data

$$v(0, y) = v_0(y). \quad (2.18)$$

Existence and uniqueness of the entropy solution $v(t, y) \in L^\infty(\mathbb{R}_+^{m+1})$ of (2.17), (2.18) follow from the analogs of Lemmas 2.3 and 2.2 for (2.17), (2.18), and it is easy to see that $v(t, y)$ is also spatially periodic, namely, $v(t, y + e_i) = v(t, y)$, for all $y \in \mathbb{R}^m$, $t > 0$, where e_j , $j = 1, \dots, m$, is the canonical basis of \mathbb{R}^m . The following assertion corresponds to theorem 2.1 of [22] and its proof follows by the same line as the proof of that result, so we just refer to [22] for the proof.

Assertion #1. For a.e. $z \in \mathbb{R}^m$ the function $u(t, x) = v(t, z + y(x))$ is an entropy solution of (1.1), (1.2) with initial data $v_0(z + y(x))$.

The next step is another observation in [22] that it follows from Birkhoff individual ergodic theorem [12] that, for any $w \in L^1(\Pi^m)$, where $\Pi^m := \mathbb{R}^m / \mathbb{Z}^m$, for almost all $z \in \Pi^m$, we have

$$\oint_{\mathbb{R}^m} w(z + y(x)) dx = \int_{\Pi^m} w(y) dy. \quad (2.19)$$

Moreover, if $w \in C(\Pi^m)$, then (2.19) holds for all $z \in \Pi^m$ and $w^z(x) := w(z + y(x))$ is a (Bohr) almost periodic function for each $z \in \Pi^m$.

The next main assertion corresponds to the first part of theorem 2.2 of [22], that is, it does not include the part about the decay of the entropy solution, and again its proof follows exactly as the one of the referred theorem and we refer to [22] for the proof. Also, in the present case we can no longer assert the continuity of the solution in t taking values in $\text{BAP}(\mathbb{R}^d)$, which is essentially based on the continuity of the periodic solution of hyperbolic problem corresponding to (2.17), (2.18), which in general is not known for the degenerate parabolic-hyperbolic equation (2.17). As we will see in the next section such continuity holds in the special case of the degenerate parabolic-hyperbolic equation (1.16), under the non-degeneracy conditions (1.17) and

(1.18). We leave the claim about the decay of the weak entropy solution to be addressed in a subsequent statement by itself.

Assertion #2. Let $u(t, x)$ be a weak entropy solution of (1.1), (1.2), assume that the initial function $u_0(x)$ is a trigonometrical polynomial with $G_0 = \text{Gr}(u_0)$. Then

$$u \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}_+^{d+1})$$

and $\text{Sp}(u(t, \cdot)) \subset G_0$ for a.e. $t > 0$.

We just observe that Assertion #2 is proved (cf. [22]) by using Assertion #1 and showing, for a suitable sequence z_l converging to 0 as $l \rightarrow \infty$, belonging to the set of full measure of $z \in \mathbb{R}^m$ given by Assertion #1, for each fixed t in a set of full measure in \mathbb{R}_+ , the convergence of the entropy solutions $u^{z_l}(t, \cdot) = v(t, z_l + y(x))$ in $\text{BAP}(\mathbb{R}^d)$, as $z_l \rightarrow 0$, uniformly with respect to t , and using that for each z_l

$$u^{z_l} \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^d)) \bigcap L^\infty(\mathbb{R}_+^{d+1})$$

and $\text{Sp}(u^{z_l}(t, \cdot)) \subset G_0$ for a.e. $t > 0$.

Now, let us consider the general case where $u_0 \in \text{BAP}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let $u(t, x)$ be the entropy solution of (1.1), (1.2) obtained above. Following [22], let $\text{Gr}(u_0)$ be the minimal additive subgroup of \mathbb{R}^d containing $\text{Sp}(u_0)$. We then consider a sequence u_{0l} of trigonometrical polynomials such that $u_{0l} \rightarrow u_0$ as $l \rightarrow \infty$, in $\text{BAP}(\mathbb{R}^d)$ and $\text{Sp}(u_{0l}) \subset \text{Gr}(u_0)$, which may be obtained from the Bochner-Fejér trigonometrical polynomials (see [2], p.105). We denote by $u_l(t, x)$ the weak entropy solution of (1.1), (1.2) with initial function $u_{0l}(x)$. By Proposition 2.1, there exists a set $F \subset \mathbb{R}_+$ of full measure such that, for all $t \in F$ and for every $l \in \mathbb{N}$, we have

$$N_1(u(t, \cdot) - u_l(t, \cdot)) \leq N_1(u_{0l} - u_0) \rightarrow 0, \quad \text{as } l \rightarrow \infty. \quad (2.20)$$

Since u_{0l} has finite spectrum, by Assertion #2 we see that $u_l(t, x) \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^d))$ and $\text{Sp}(u_l(t, \cdot)) \subset \text{Gr}(u_0)$, for all $t \in F$, for all $l \in \mathbb{N}$. Therefore, $u \in L^\infty([0, \infty), \text{BAP}(\mathbb{R}^d))$. Moreover, we easily see that $\text{Sp}(u(t, \cdot)) \subset \text{Gr}(u_0)$, for a.e. $t > 0$. □

We now turn to the proof of the decay property (1.15).

Lemma 2.5 *The entropy solution of (1.1), (1.2), with initial function u_0 verifying (1.5), satisfies (1.15).*

PROOF: Again, reasoning as in [22], consider first the case where u_0 is a trigonometrical polynomial as in (2.14) and (2.15), which may be a Bochner-Fejér trigonometric polynomial for the original initial function, and let $v(t, y)$

be the entropy solution of (2.17),(2.18). Writing $(\mathcal{B}\nabla_y) \cdot (A(v)\nabla_y v)$ in the form $\nabla_y(\tilde{B}(v)\nabla_y v)$, we have, for any vector $\kappa \in \mathbb{R}^m$, we have

$$\kappa^\top \tilde{B}(\xi)\kappa = (\mathcal{B}\tau)^\top A(\xi)\mathcal{B}\tau.$$

Also,

$$\tilde{\mathbf{f}}'(v) \cdot \kappa = \mathbf{f}'(v) \cdot \mathcal{B}\kappa.$$

Therefore, we see that conditions (1.5) implies that, for all $(\tau, \kappa) \in \mathbb{R}^{m+1}$, with $\tau^2 + \kappa^2 = 1$,

$$\mathcal{L}^1\{\xi \in \mathbb{R} : |\xi| \leq \|v_0\|_\infty, \tau + \tilde{\mathbf{f}}'(\xi) \cdot \kappa = 0, \kappa^\top \tilde{B}(v)\kappa = 0\} = 0. \quad (2.21)$$

Hence, we can apply the theorem of Chen and Perthame in [10] to obtain that

$$\lim_{t \rightarrow \infty} \int_{\Pi^m} |v(t, y) - M(v_0)| dy = 0, \quad (2.22)$$

where

$$M(v_0) = \int_{\Pi^m} v_0(y) dy = \int_{\mathbb{R}^d} u_0(x) dx = M(u_0).$$

Now, we use the fact that

$$\int_{\mathbb{R}^d} |u(t, x) - C| dx = \int_{\Pi^m} |v(t, y) - C| dy, \quad (2.23)$$

for any $C \in \mathbb{R}$, which follows by using (2.20) and an approximation argument, for which we refer to [22]. Taking $C = M(u_0)$ and using (2.22), we conclude that (1.15) holds in the case where u_0 is a trigonometric polynomial.

The general case follows easily from the case just proved by using the Bochner-Fejér approximation by trigonometric polynomials, which finishes the proof. □

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. This amounts to proving the strong trace property for the weak entropy solution of (1.16),(1.2), at all hyperplane $t = t_0$, for all $t_0 \geq 0$. Indeed, by the Gauss-Green Theorem (see, e.g., [6], [7]), applied to the (divergence-free) L^2 -divergence-measure field $(u, \mathbf{f}(u) - \nabla_{x''} b(u))$, we easily deduce that the limits $\lim_{t \rightarrow t_0 \pm} u(t, x)$ exist in the weak star topology of $L^\infty(\mathbb{R}^d)$, for $t_0 > 0$, and just the limit for t_{0+} when $t_0 = 0$. By the same result, for $t_0 > 0$, using the fact that the referred field is divergence-free, we easily deduce that the limits for t_{0+} and t_{0-} must coincide. We also refer to theorem 4.5.1 of [11] whose proof establishes the continuity of $u(t, \cdot)$ from $(0, \infty)$ into $L^1_{\text{loc}}(\mathbb{R}^d)$ except for a countable set of $t \in (0, \infty)$. As observed in

[11], the continuity at t_0 would follow if the entropy inequality included the initial time, which is not the case here, where we consider the weak initial prescription (1.19).

We rewrite (1.8) for the present case. For any convex C^2 function $\eta : \mathbb{R} \rightarrow \mathbb{R}$, and $\mathbf{q}'(u) = \eta'(u)\mathbf{f}(u)$, $r'_{ij}(u) = \eta'(u)b_{ij}(u)$, $i, j = d' + 1, \dots, d$, we have

$$\partial_t \eta(u) + \nabla_x \cdot \mathbf{q}(u) - \sum_{i,j=d'+1}^d \partial_{x_i x_j}^2 r_{ij}(u) \leq -\eta''(u) \sum_{k=d'+1}^d \left(\sum_{i=d'+1}^d \partial_{x_i} \beta_{ik}(u) \right)^2, \quad (3.1)$$

in the sense of distributions in $(0, \infty) \times \mathbb{R}^d$, where $\beta_{ik}(u) = \int^u \sigma_{ik}(v) dv$ and $\Sigma(u) = (\sigma_{ij}(u))_{i,j=d'+1}^d$ satisfies $B(u) = \Sigma(u)^2$.

We will use the kinetic formulation for (1.1) (cf. [9]). So, we introduce the kinetic function χ on \mathbb{R}^2 :

$$\chi(\xi; u) = \begin{cases} 1 & \text{for } 0 < \xi < u, \\ -1 & \text{for } u < \xi < 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following representation holds for any $S \in C^1(\mathbb{R})$,

$$S(u) = \int_{\mathbb{R}} S'(\xi) \chi(\xi; u) d\xi, \quad (3.2)$$

which yields the following kinetic equation equivalent to (3.1):

$$\partial_t \chi(\xi; u) + \mathbf{a}(\xi) \cdot \nabla_x \chi(\xi; u) - \sum_{i,j=d'+1}^d b_{ij}(\xi) \partial_{x_i x_j}^2 \chi(\xi; u) = \partial_\xi (m+n)(t, x, \xi) \quad (3.3)$$

in the sense of distributions in $(0, \infty) \times \mathbb{R}^{d+1}$.

In (3.3), $m(t, x, \xi)$, $n(t, x, \xi)$ are non-negative measures satisfying, for all $R, T > 0$,

$$\int_{C_{R,T}} (m+n)(t, x, \xi) dx dt \leq \mu_{R,T}(\xi) \in L_0^\infty(\mathbb{R}), \quad (3.4)$$

where $C_{R,T} = (0, T) \times C_R$, and by L_0^∞ we mean L^∞ with compact support, and

$$n(t, x, \xi) = \delta(\xi - u(t, x)) \sum_{k=d'+1}^d \left(\sum_{i=d'+1}^d \partial_{x_i} \sigma_{ik}(u(t, x)) \right)^2. \quad (3.5)$$

Also, taking $\eta(u) = \frac{1}{2}u^2$ in (1.8), we see that

$$\int_{\mathbb{R}} \int_{C_{R,T}} (m+n)(t, x, \xi) dx dt d\xi \leq C(R, T), \quad (3.6)$$

for all $R, T > 0$, for some constant $C(R, T) > 0$ depending only on R, T and $\|u_0\|_\infty$.

Equation (3.1) implies that for any convex entropy η , the vector field $F = (\eta(u), \mathbf{q}(u) - (\sum_{i=1}^d \partial_{x_i} \hat{r}_{ij}(u))_{j=1}^d) \in \mathcal{DM}^2(C_{R,T})$, where

$$\hat{r}_{ij}(u) = \begin{cases} 0, & \text{for } 1 \leq i \leq d' \text{ or } 1 \leq j \leq d', \\ r_{ij}(u), & \text{for } d' + 1 \leq i, j \leq d \end{cases}$$

and for any $R > 0, T > 0$, that is, it is an L^2 divergence-measure field on $C_{R,T}$. By theorems 3.1 and 3.2 in [15], or essentially also from lemma 1.3.3 in [11], the normal trace of the \mathcal{DM}^2 -field F at the hyperplane $t = t_* \in (0, T)$, from above, that is, as a part of the boundary of $C_{R,T} \cap \{t > t_*\}$, as well as from below, that is, as part of the boundary of $C_{R,T} \cap \{t < t_*\}$, is simply given by

$$\langle F \cdot \nu, \phi \rangle_{t=t_* \pm} = \int_{\mathbb{R}^d} \eta(u(t_*, x)) \phi(x) dx,$$

for a.e. $t_* > 0$, for any $\phi \in C_c^1(\mathbb{R}^d)$, where $\langle F \cdot \nu, \cdot \rangle_{t=t_*+}$ denotes the normal trace at $\{t = t_*\}$ from above and $\langle F \cdot \nu, \cdot \rangle_{t=t_*-}$ the one from below. Also, from theorem 3.2 in [15] or also essentially from lemma 1.3.3 in [11], we deduce that, for any $t_0 > 0$,

$$\langle F \cdot \nu, \phi \rangle_{t=t_0 \pm} = \text{ess} \lim_{t \rightarrow t_0 \pm} \int_{\mathbb{R}^d} \eta(u(t, x)) \phi(x) dx, \quad (3.7)$$

for any $\phi \in C_c^1(\mathbb{R}^d)$, and for $t_0 = 0$ we have, similarly,

$$\langle F \cdot \nu, \phi \rangle_{t=0+} = \text{ess} \lim_{t \rightarrow 0+} \int_{\mathbb{R}^d} \eta(u(t, x)) \phi(x) dx. \quad (3.8)$$

Now, using (3.1) and the representation (3.2) for an arbitrary convex η , we deduce that, for $f(t, x, \xi) = \chi(\xi; u(t, x))$, there exists the limit

$$\lim_{t \rightarrow t_0+} f(t, \cdot, \cdot) = f^\tau(\cdot, \cdot), \quad (3.9)$$

in the weak star topology of $L^\infty(C_R \times (-L, L))$, for any $R > 0$, and any $L > 0$ satisfying $\|u\|_{L^\infty(\mathbb{R}_+^{d+1})} \leq L$. Similarly, we have

$$\lim_{t \rightarrow t_0-} f(t, \cdot, \cdot) = f^- (\cdot, \cdot), \quad (3.10)$$

in the weak star topology of $L^\infty(C_R \times (-L, L))$. We observe that for $\eta(u) = u$, for all $t_0 > 0$,

$$\text{ess} \lim_{t \rightarrow t_0+} \int_{\mathbb{R}^d} u(t, x) \phi(x) dx = \text{ess} \lim_{t \rightarrow t_0-} \int_{\mathbb{R}^d} u(t, x) \phi(x) dx, \quad (3.11)$$

for all $\phi \in C_c^1(\mathbb{R}^d)$, as a consequence of (3.7), (3.8) and the Gauss-Green formula [6, 7] (essentially also from lemma 1.3.3 in [11]). Therefore, if the existence of strong trace of $u(t, x)$ at $t = t_0$ can be proved, both from above and below, these strong traces must coincide. Since the proof of the strong trace property from below is totally analogous to that for the strong trace from above, it will suffice to investigate the latter.

Following the method in [26], in order to prove that the limits in (3.9) and (3.10) can be taken as the strong convergence in $L^1(C_{R,T} \times (-L, L))$, it suffices to prove that $f^\tau(\cdot, \cdot)$ is a χ -function, which is proved by using localization method introduced in [26]. For simplicity we just consider the case $t_0 = 0$.

We write for $x \in \mathbb{R}^d$, $x = (x', x'')$, where $x' \in \mathbb{R}^{d'}$, $x'' \in \mathbb{R}^{d''}$. Fixing, $x_0 \in \mathbb{R}^d$, we consider the sequence

$$f_\varepsilon(\underline{t}, \underline{x}, \xi) := f(\varepsilon \underline{t}, x_0 + \Lambda(\varepsilon) \underline{x}, \xi),$$

where $\Lambda(\varepsilon) \underline{x} = (\varepsilon \underline{x}', \varepsilon^{1/2} \underline{x}'')$. So, f_ε satisfies

$$\partial_{\underline{t}} f_\varepsilon + \mathbf{a}(\xi)' \cdot \nabla_{\underline{x}'} f_\varepsilon + \varepsilon^{1/2} \mathbf{a}(\xi)'' \cdot \nabla_{\underline{x}''} f_\varepsilon - \sum_{i,j=d'+1}^d b_{ij}(\xi) \partial_{\underline{x}_i \underline{x}_j}^2 f_\varepsilon = \partial_\xi (m_\varepsilon + n_\varepsilon), \quad (3.12)$$

where $\mathbf{a}(\xi)' = (\pi_{d'}(\mathbf{a}(\xi)), \underbrace{0, \dots, 0}_{d''})$, $\mathbf{a}(\xi)'' = \mathbf{a}(\xi) - \mathbf{a}(\xi)'$, and $m_\varepsilon \in \mathcal{M}_{\text{loc}}^+((0, \infty) \times \mathbb{R}^d \times \mathbb{R})$ is defined, for every $0 \leq R_1^0 < R_2^0$, $R_1^i < R_2^i$, $i = 1, \dots, d$, $L_1 < L_2$, by

$$\begin{aligned} & (m_\varepsilon + n_\varepsilon) \left(\prod_{0 \leq i \leq d} [R_1^i, R_2^i] \times [L_1, L_2] \right) \\ &= \frac{1}{\varepsilon^{d'+\frac{d''}{2}}} (m + n) \left([\varepsilon R_1^0, \varepsilon R_2^0] \times (x_0 + \Lambda(\varepsilon) \prod_{1 \leq i \leq d} [R_1^i, R_2^i]) \times [L_1, L_2] \right), \end{aligned}$$

where $\Lambda(\varepsilon) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by $\Lambda(\varepsilon)z := (\varepsilon z', \varepsilon^{1/2} z'')$.

Following [26], as in [16], we have there exists a sequence ε_n converging to 0 and a set $\mathcal{E} \subset \mathbb{R}^d$, with $\mathcal{L}^d(\mathbb{R}^d \setminus \mathcal{E}) = 0$, such that for all $x_0 \in \mathcal{E}$

$$\lim_{\varepsilon \rightarrow 0} (m_{\varepsilon_n} + n_{\varepsilon_n}) = 0, \quad (3.13)$$

in the weak topology of $\mathcal{M}_{\text{loc}}^+((0, \infty) \times \mathbb{R}^d \times \mathbb{R})$.

We now observe that

$$f_\varepsilon(0, \underline{x}, \xi) = f^\tau(x_0 + \Lambda(\varepsilon) \underline{x}, \xi). \quad (3.14)$$

Again following [26], as in [16], we have that there exists a subsequence still denoted ε_n and a subset \mathcal{E}' of \mathbb{R}^d such that for every $x_0 \in \mathcal{E}'$ and for every

$R > 0$,

$$\lim_{\varepsilon_n \rightarrow 0} \int_{-L}^L \int_{(-R, R)^d} |f^\tau(x_0, \xi) - f^\tau(x_0 + \Lambda(\varepsilon_n)\underline{x}, \xi)| d\underline{x} d\xi = 0. \quad (3.15)$$

Now, we claim that there exists a sequence ε_n which goes to 0 and a χ -function $f_\infty \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times (-L, L))$ such that f_{ε_n} converges strongly to f_∞ in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^d \times (-L, L))$ and

$$\partial_{\underline{t}} f_\infty + \mathbf{a}(\xi)' \cdot \nabla_{\underline{x}'} f_\infty - \sum_{i,j=d'+1}^d b_{ij}(\xi) \partial_{\underline{x}_i \underline{x}_j}^2 f_\infty = 0. \quad (3.16)$$

The proof of the claim is very similar to that of proposition 3 of [26], and lemma 3.1 in [16], and relies on a particular case of the version of averaging lemma in [24] (see also [25]). Here, we need the following variation of the standard averaging lemma.

Lemma 3.1 *Let N, N', N'' be positive integers with $N = N' + N''$, $f_n(y, \xi)$ be a bounded sequence in $L^2(\mathbb{R}^N \times \mathbb{R}) \cap L^1(\mathbb{R}^N \times \mathbb{R})$, $\mathbf{g}_n^i, \mathbf{g}^i \in L^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{N+1})$ be such that $\mathbf{g}_n^i \rightarrow \mathbf{g}^i$ strongly in $L^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^{N+1})$, $i = 1, 2$, and for $y \in \mathbb{R}^N$ we write $y = (y', y'')$, $y' \in \mathbb{R}^{N'}$, $y'' \in \mathbb{R}^{N''}$. Assume*

$$\alpha(\xi)' \cdot \nabla_{y'} f_n + \alpha(\xi)'' \cdot \nabla_{y''} f_n - \sum_{i,j=d'+1}^d \beta_{ij}(\xi) \partial_{y_i y_j}^2 f_n = \partial_\xi \nabla_{y, \xi} \cdot \mathbf{g}_n^1 + \nabla_{y, \xi} \cdot \mathbf{g}_n^2, \quad (3.17)$$

where $\alpha(\cdot)' \in C^2(\mathbb{R}; \mathbb{R}^{N'})$, $\alpha(\cdot)'' \in C^2(\mathbb{R}; \mathbb{R}^{N''})$, $\beta \in C^2(\mathbb{R})$ satisfy

$$\mathcal{L}^1 \{ \xi \in \mathbb{R} : \alpha(\xi) \cdot \zeta' = 0 \} = 0, \quad \text{for every } \zeta' \in \mathbb{R}^{N'}, \text{ with } |\zeta'| = 1, \quad (3.18)$$

where \mathcal{L}^1 is the Lebesgue measure on \mathbb{R} , and also

$$\mathcal{L}^1 \{ \xi \in \mathbb{R} : \beta(\xi) = 0 \} = 0. \quad (3.19)$$

Then, for any $\phi \in C_c^\infty(\mathbb{R})$, the average $u_n^\phi(y) = \int_{\mathbb{R}} \phi(\xi) f_n(y, \xi) d\xi$ is relatively compact in $L^2(\mathbb{R}^N)$.

The application of Lemma 3.1 to the problem at hand is made, as in [26], by multiplying (3.12) by $\phi_1(\underline{t}, \underline{x}), \phi_2(\xi)$ where $\phi_1 \in C_0^\infty((1/(2R), 2R) \times (-2R, 2R)^d)$, $\phi_2 \in C_0^\infty(-2L, 2L)$, both taking values in $[0, 1]$, with $\phi_1(\underline{t}, \underline{x}) = 1$, for $(\underline{t}, \underline{x}) \in (1/R, R) \times (-R, R)^d$, $\phi_2(\xi) = 1$, for $\xi \in (-L, L)$. We then consider the equation obtained for $\phi_1 \phi_2 f_\varepsilon$, which is easily seen to satisfy the hypotheses of Lemma 2.1, we refer to [26] for the details.

The final step of the proof is to proof that for every $x_0 \in \mathcal{E}'$,

$$f_\infty(0, \underline{x}, \xi) = f^\tau(x_0, \xi), \quad (3.20)$$

for a.e. $(\underline{x}, \xi) \in \mathbb{R}^d \times (-L, L)$, which the result corresponding to proposition 4 of [26]. The proof is the same as the one of the referred proposition, and consists in proving that, for any $\phi \in C_0^\infty(\mathbb{R}^d \times (-L, L))$, the sequence

$$h_\phi^\varepsilon(t) := \int_{-L}^L \int_{\mathbb{R}^d} (f_\varepsilon(\underline{t}, \underline{x}, \xi) - f_\infty(\underline{t}, \underline{x}, \xi)) \phi(\underline{x}, \xi) d\underline{x} d\xi,$$

converges to 0 in $BV((0, 1))$, which is done exactly as in [26].

Finally, from (3.16) and (3.20), we easily conclude that

$$f_\infty(\underline{t}, \underline{x}, \xi) = f^\tau(x_0, \xi),$$

for almost all $(\underline{t}, \underline{x}, \xi) \in \mathbb{R}^{d+1} \times (-L, L)$, which is constant with respect to $(\underline{t}, \underline{x})$. Hence, since f_∞ is a χ -function for almost all $(\underline{t}, \underline{x})$, we conclude that $f^\tau(x_0, \cdot)$ is a χ -function, as was to be proved. The proof of the strong trace property at any hyperplane $\{t = t_0\}$, $t_0 > 0$, both from above and from below, follows exactly as just done for $t_0 = 0$, from above, and this concludes the first part of proof of Theorem 1.2.

As for the last part of Theorem 1.2, concerning (1.21), it follows from the fact that in the proof of Lemma 2.4 the periodic solution of the equation corresponding to (2.17), with initial function (2.18), $v(t, y)$, satisfies $v \in C([0, \infty), L_{\text{loc}}^1(\mathbb{R}^d))$ as a consequence of the first part, already proved, of Theorem 1.2. Since the convergence of $u^l(t, \cdot) = v(t, z_l + y(\cdot))$, as $l \rightarrow \infty$, in $\text{BAP}(\mathbb{R}^d)$, in uniform in time, we have that $u \in C([0, \infty), \text{BAP}(\mathbb{R}^d))$, when u_0 is a trigonometric polynomial. Also, for a general initial data, by (2.20), the convergence of the solutions corresponding to initial trigonometric polynomial, approximating the initial function, is uniform in time with values in $\text{BAP}(\mathbb{R}^d)$, so we conclude that $u \in C([0, \infty), \text{BAP}(\mathbb{R}^d))$. The fact that $\text{Sp}(u(t, x)) \subset \text{Gr}(u_0)$ for all $t > 0$, follows from Lemma 2.4 and the continuity just proved.

4. Alternative proof of the decay property

In this section we give an alternative proof of the decay property for the (weak) solution of (1.16), (1.2), using Theorem 1.2.

Now, as a final remark, we show that the decay property may also be proved using some ideas in [14]. First, as a consequence of the fact that $\text{Gr}(u(t, x)) \subset \text{Gr}(u_0)$, we have the following result. We recall that the space of Stepanoff almost periodic functions (with exponent $p = 1$) in \mathbb{R}^d , $\text{SAP}(\mathbb{R}^d)$, is defined as the completion of the trigonometric polynomials with respect to the norm

$$\|f\|_S := \sup_{x \in \mathbb{R}^r} \int_{C_1(x)} |f(y)| dy = \sup_{x \in \mathbb{R}^d} \int_{C_1} |f(y+x)| dy,$$

where

$$C_R(x) := \{y \in \mathbb{R}^d : |y - x|_\infty := \max_{i=1, \dots, d} |y_i - x_i| \leq R/2\}.$$

Another characterization of the Stepanoff almost periodic function (S-a.p., for short) is obtained introducing the concept of ε -period of a function f , that is a number τ satisfying

$$\|f(\cdot + \tau) - f(\cdot)\|_S \leq \varepsilon. \quad (4.1)$$

Let $E_S\{\varepsilon, f\}$ denote the set of such numbers. If the set $E_S\{\varepsilon, f\}$ is relatively dense for all positive values of ε , then the function f is S-a.p. (see, e.g., [2]). By the set $E_S\{\varepsilon, f\}$ being relatively dense it is meant that there exists a length l_ε , called *inclusion interval*, such that for any $x \in \mathbb{R}^d$, $C_{l_\varepsilon}(x)$ contains an element of $E_S\{\varepsilon, f\}$. The following lemma is of interest in its own.

Lemma 4.1 *If u_0 is a trigonometric polynomial, then the weak entropy solution of (1.16), (1.2), $u(t, x)$, is S-a.p. for all $t > 0$, and for any $\varepsilon > 0$, there exists $l_\varepsilon > 0$ which is an inclusion interval for $u(t, \cdot)$, for all $t > 0$.*

PROOF: Clearly, u_0 , being a trigonometric polynomial, is S-a.p. The fact that $u(t, x)$ is S-a.p. for all $t > 0$ follows from (2.9), with $v(t, x) = u(t, x + \tau)$ and $\rho(x - x_0)$ instead of $\rho(x)$, from which we deduce

$$\begin{aligned} \int_{C_1(x_0)} |u(t, x + \tau) - u(t, x)| dx &\leq c(t) \int_{C_R(x_0)} |u_0(x) - u_0(x + \tau)| \rho(x - x_0) dx \\ &+ O\left(\frac{1}{R}\right) \leq c(R, t) \sup_{x \in \mathbb{R}^d} \int_{C_1(x)} |u_0(y + \tau) - u_0(y)| dy + O\left(\frac{1}{R}\right), \end{aligned} \quad (4.2)$$

where $c(R, t)$ is a positive constant depending only on R, t and $O(\frac{1}{R})$ goes to zero when $R \rightarrow \infty$ uniformly with respect to x_0 . So, choosing R large enough so that $O(1/R) \leq \varepsilon/2$ and then taking any $\tau \in E_S\{\varepsilon/(2c(R, t)), u_0\}$, we get that $\tau \in E_S\{\varepsilon, u(t, \cdot)\}$, and so $u(t, \cdot)$ is S-a.p.

Now we prove that for any $\varepsilon > 0$, there exists $l_\varepsilon > 0$ which is an inclusion interval for $u(t, \cdot)$, for all $t > 0$. For this we use the following two results of [2], p. 53, which hold for the classical almost periodic functions (u.a.p., for short) in the line. The ε -periods of a u.a.p. function f are defined by (4.1) with $\|\cdot\|_S$ replaced by the sup-norm $\|\cdot\|_\infty$ and we denote the set of such ε -periods by $E\{\varepsilon, f\}$.

Assertion #1. Given a u.a.p. function

$$f(x) \sim \sum_{n=1}^{\infty} A_n e^{i\Lambda_n x}, \quad (4.3)$$

to any positive integer N and a positive number $\delta < \pi$ corresponds a positive ε such that all numbers τ of the set $E\{\varepsilon, f(x)\}$ satisfy the following Diophantine inequalities:

$$|\Lambda_n \tau| < \delta \pmod{2\pi}, \quad n = 1, \dots, N, \quad (4.4)$$

where the inequality means that there exists $k \in \mathbb{Z}$ such that $|\Lambda_n \tau - 2\pi k| < \delta$.

Assertion #2. Given a u.a.p. function as in (4.3), to any $\varepsilon > 0$ corresponds a positive integer N and a positive $\delta < \pi$ such that any number τ satisfying the N Diophantine inequalities (4.4), belongs to $E\{\varepsilon, f(x)\}$.

These assertions are stated for u.a.p. functions in \mathbb{R} , but it is easy to see that they can be easily extended to u.a.p. functions in \mathbb{R}^d . Now, let us show how these two assertions from [2] can be applied to prove the lemma. For any $t > 0$, we approximate $u(t, x)$ by the corresponding Bochner-Fejér trigonometric polynomials, whose spectrum is contained in $\text{Sp}(u(t, \cdot))$ and the coefficients have absolute values dominated by the corresponding coefficient in $u(t, \cdot)$, which, in turn, are dominated by those of the initial function by Proposition 2.1. Therefore, Assertion #2, whose prove depends only on the frequencies and absolute value of the coefficients (cf. [2]), implies that the set of τ 's satisfying (4.4) is contained in $E\{\varepsilon, u(t, \cdot)\}$, for all $t > 0$. On the other hand, Assertion #1 implies that the set of such τ 's satisfying (4.4) is relatively dense, since it contains $E\{\varepsilon', u_0\}$ for some $\varepsilon' > 0$, and such sets are relatively dense. Therefore, we can find $l_\varepsilon > 0$ which is an inclusion interval for $u(t, \cdot)$ for all $t > 0$, which was to be proved. □

Now we can use Lemma 4.1 to give an alternative proof of the decay property (1.15). Clearly, from Proposition 2.1, by approximating the initial function by Bochner-Fejér trigonometric polynomials, it suffices to consider the case where the initial function u_0 is itself a trigonometric polynomial. Let us then consider the scaling sequence $u^T(t, x) := u(Tt, Tx', \sqrt{T}x'')$, and define $\xi' = x'/t$, $\xi'' = x''/\sqrt{t}$. So, u^T is a uniformly bounded sequence of weak entropy solutions of (1.1), (1.2), with initial functions $u_0^T(x) := u_0(Tx', \sqrt{T}x'')$. Using the Averaging Lemma 3.1, we deduce that u^T is relatively compact in $L_{\text{loc}}^1(\mathbb{R}_+^{d+1})$ and the initial functions clearly converge weakly to $\bar{u}_0 = M(u_0)$. By passing to a subsequence that we still denote by $u^T(t, x)$, we have that $u^T \rightarrow \bar{u}$ as $T \rightarrow \infty$, in $L_{\text{loc}}^1(\mathbb{R}^{d+1})$, for some $u \in L^\infty(\mathbb{R}_+^{d+1})$. We see also that \bar{u} satisfies (1.6), (1.7), (1.8) and (1.19), all of which are easy to be verified, and we observe by (1.19) that $\bar{u}(0, x) = M(u_0)$. Now, in view of Theorem 1.2, by the uniqueness Lemma 2.2, we conclude that $\bar{u}(t, x) = M(u_0)$, that is $u^T \rightarrow M(u_0)$,

in $L^1_{\text{loc}}(\mathbb{R}^{d+1})$. This, in particular, implies

$$\begin{aligned}
0 &= \lim_{T \rightarrow \infty} \int_0^1 \int_{|x'| \leq c', |x''| \leq c''} |u(Tt, Tx', \sqrt{T}x'') - M(u_0)| dx' dx'' dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{1}{T^{d'+d''/2}} \int_{|x'| \leq c'T, |x''| \leq c''\sqrt{T}} |u(t, x', x'') - M(u_0)| dx' dx'' dt \\
&\geq \frac{1}{2^{d'+d''/2}} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{T/2}^T \int_{|\xi'| \leq c', |\xi''| \leq c''} |u(t, \xi't, \xi''\sqrt{t}) - M(u_0)| d\xi' d\xi'' dt,
\end{aligned}$$

which implies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_{|\xi'| \leq c', |\xi''| \leq c''} |u(t, \xi't, \xi''\sqrt{t}) - M(u_0)| d\xi' d\xi'' dt = 0, \quad (4.5)$$

as is easily seen. Now, invoking Lemma 4.1, we can then make a computation similar to that in p.51 of [14] in order to get, for all $t > 0$ large enough,

$$\int_{|\xi'| \leq c', |\xi''| \leq c''} |u(t, \xi't, \xi''\sqrt{t}) - M(u_0)| d\xi' d\xi'' \geq c_1 M(|u(t, \cdot) - M(u_0)|), \quad (4.6)$$

for certain positive constant c_1 depending only on the dimension. Therefore, by (4.5), we deduce

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T M(|u(t, \cdot) - M(u_0)|) dt = 0. \quad (4.7)$$

Now, by Proposition 2.1, we conclude

$$\lim_{t \rightarrow \infty} M(|u(t, \cdot) - M(u_0)|) dt = 0, \quad (4.8)$$

which is the desired result.

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References

- [1] Bendahmane, M., Karlsen, K. *Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations*. SIAM J. Math. Anal. **36** (2004), No. 2, 405–422.

- [2] Besicovitch, A.S. “Almost Periodic Functions”. Cambridge University Press, 1932.
- [3] Carrillo, J. *Entropy solutions for nonlinear degenerate problems*. Arch. Rat. Mech. Anal. **147** (1999), 269–361.
- [4] Chen, G.-Q., Frid, H. *Decay of entropy solutions of nonlinear conservation laws*. Arch. Rational Mech. Anal. **146** (1999), No.2, 95–127.
- [5] Chen, G.-Q., Frid, H. *Divergence-measure fields and hyperbolic conservation laws*. Arch. Ration. Mech. Anal. **147** (1999), no. 2, 89 –118.
- [6] Chen, G.-Q., Frid, H. *On the theory of divergence-measure fields and its applications*. Bol. Soc. Brasil. Mat. (N.S.) **32** (2001), no. 3, 401–433.
- [7] Chen, G.-Q., Frid, H. *Extended divergence-measure fields and the Euler equations for gas dynamics*. Comm. Math. Phys. **236** (2003), no. 2, 251–280.
- [8] Chen, G.-Q., Karlsen, K.H. *Quasilinear anisotropic degenerate parabolic equations with time-space dependent diffusion coefficients*. Communications On Pure and Applied Analysis, **4**, Number 2, 2005, 241–266.
- [9] Chen, G.-Q., Perthame, B. *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations*. Ann. I. H. Poincaré, **20** (2003), 645–668.
- [10] Chen, G.-Q., Perthame, B. *Large-time behavior of periodic entropy solutions to anisotropic degenerate parabolic-hyperbolic equations*. Proc. American Math. Soc. **137**, No. 9 (2009), 3003–3011.
- [11] Dafermos, C.M. “Hyperbolic Conservation Laws in Continuum Physics” (Third Edition). Springer-Verlag, Berlin, Heidelberg, 1999, 2005, 2010.
- [12] Dunford, N., Schwartz, J.T. “Linear Operators. General Theory, Part I ”. Interscience Publishers, Inc., New York, 1958, 1963.
- [13] Endal, J., Jakobsen, E.R. *L^1 contraction for bounded (nonintegrable) solutions of degenerate parabolic equations*. SIAM J. Math. Anal. **46** (2014), no. 6, 3957–3982.
- [14] Frid, H. *Decay of almost periodic solutions of conservation laws*. Arch. Rational Mech. Anal. **161** (2002), 43–64.
- [15] Frid, H. *Divergence-measure fields on domains with Lipschitz boundary*. “Hyperbolic conservation laws and related analysis with applications”, 207–225, Springer Proc. Math. Stat., **49**, Springer, Heidelberg, 2014.
- [16] Frid, H., Li, Y. *A boundary value problem for a class of anisotropic degenerate parabolic-hyperbolic equations*. Arxiv preprint (2016), in <http://arxiv.org/abs/1606.05795>.
- [17] Karlsen, K.H., Risebro, N.H. *On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients*. Discrete Contin. Dyn. Syst. **9** (2003), no. 5, 1081–1104.
- [18] Kruzhkov, S.N. *First order quasilinear equations in several independent variables*. Math. USSR-Sb. **10** (1970), 217–243.
- [19] Ladyzhenskaya, O.A., Solonnikov, V.A., Ural’ceva, N.N. “Linear and Quasilinear Equations of Parabolic Type”, Providence, R.I.: Amer. Math. Soc. 1968.

- [20] Mascia, C., Porreta, A., Terracina, A. *Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations*. Arch. Rational Mech. Anal. **163** (2002), 87–124.
- [21] Panov, E. *Existence of strong traces for quasi-solutions of multidimensional conservation laws*. Journal of Hyperbolic Differential Equations Vol. 4, No. 4 (2007), 729–770.
- [22] Panov, E. *On the Cauchy problem for scalar conservation laws in the class of Besicovitch almost periodic functions: global well-posedness and decay property*. Journal of Hyperbolic Diff. Equations, Vol. 13, No. 3 (2016), 633–659.
- [23] Perthame, B. “Kinetic formulations of parabolic and hyperbolic PDEs: from theory to numerics. Evolutionary equations”. Vol. I, 437–471, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.
- [24] Perthame, B., Souganidis, P.E. *A limiting case for velocity averaging*. Ann. Sci. Ecole Norm. Sup. (4) **31** (1998), 591–598.
- [25] Tadmor, E., Tao, T. *Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs*. Comm. Pure Appl. Math. **LX** (2007), 1488–1521.
- [26] Vasseur, A. *Strong traces for solutions of multidimensional scalar conservation laws*. Arch. Ration. Mech. Anal. **160** (2001) 181–193.
- [27] Vol’pert, A.I., Hudjaev, S.I. *Cauchy’s problem for degenerate second order quasi-linear parabolic equations*. Math. USSR Sbornik **7** (1969), No. 3, 365–387.

Instituto de Matemática Pura e Aplicada-IMPA
Estrada Dona Castorina, 110, CEP 22460-320, Rio de Janeiro, RJ, Brazil
E-mail: hermano@impa.br